Reference results for the multivariate normal distribution Let Y and μ be $n\times 1$ vectors and Σ be an $n\times n$ positive definite matrix. The distribution of Y is called multivariate normal, denoted as $Y \sim N(\mu, \Sigma)$, and its PDF is given by

$$f_Y(y) = (2\pi)^{-n/2} (\mathsf{det}(\Sigma))^{-1/2} \exp\left[-\frac{1}{2}(y-\mu)' \Sigma^{-1} (y-\mu)\right].$$

The usual normal distribution is obtained by setting $n = 1$, while the bivariate distribution is obtained by setting $n = 2$.

Some more notation for partitions of Y, μ, and Σ can be very useful:

$$Y = \begin{pmatrix} \underset{(n_1\times 1)}{Y_1} \\ \underset{(n_2\times 1)}{Y_2} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Take note that $\Sigma_{12} = \Sigma_{21}'$.

The following are properties of the multivariate normal:

1. $\mathbb{E}(Y) = \mu$ and $\mathsf{V}(Y) = \Sigma$.

2. The marginal distribution of Y_1 is also multivariate normal, i.e., $Y_1 \sim N(\mu_1, \Sigma_1)$.

3. The conditional distribution of $Y_2|Y_1$ is also multivariate normal, i.e.,

   $$Y_2|Y_1 = y_1 \sim N(\alpha + B'y_1, \Sigma_{22} - B'\Sigma_{11}B),$$

   where $\alpha = \mu_2 - B'\mu_1$ and $B = \Sigma_{11}^{-1}\Sigma_{12}$. Relate this result to the our discussion on conditional prediction.

4. Recall that independence implies uncorrelatedness. But under normality, we have the reverse implication, i.e., if $\Sigma_{12} = 0$, then Y_1 and Y_2 are independent random vectors.

5. Linear functions of Y are also multivariate normal, i.e., if $\underset{(m\times 1)}{Z} = \underset{(m\times 1)}{G} + \underset{(m\times n)}{H} Y$, where G and H are nonstochastic and H is full row rank, then $Z \sim N(G + H\mu, H\Sigma H')$. If you set $H = \Sigma^{-1/2}$ and $G = -\Sigma^{-1/2}\mu$, then you produce a standard normal random vector.

6. A particular quadratic function of Y is distributed as χ^2, i.e., $(Y-\mu)' \Sigma^{-1} (Y-\mu) \sim \chi^2_n$.

7. Suppose $Z \sim N(0, I_n)$. Let M be a nonstochastic $n\times n$ symmetric idempotent matrix with $\mathsf{rank}(M) = r \le n$. Then $Z'MZ \sim \chi^2_r$.

8. Suppose $Z \sim N(0, I_n)$. Let M be a nonstochastic $n\times n$ symmetric idempotent matrix with $\mathsf{rank}(M) = r \le n$. Let L be a nonstochastic matrix such that $LM = 0$. Note that there are no other restrictions on L aside from the requirement that LM should be well-defined. Then, MZ and LZ are independent random vectors.

9. Let $W_1 \sim \chi^2_m$ and $W_2 \sim \chi^2_n$. Further assume that W_1 and W_2 are independent. Then

   $$\frac{W_1/m}{W_2/n} \sim F_{m,n}.$$

   A special case is the following: Let $Z \sim N(0,1)$ (or $Z = \sqrt{W_1} \sim N(0,1)$) and $W_2 \sim \chi^2_n$. Further assume that Z and W_2 are independent. Then $\sqrt{W_1/1}/\sqrt{W_2/n} = Z/\sqrt{W_2/n} \sim t_n$.