Detour: Inequalities (from Hansen's lecture notes)

1. (Triangle inequality) For any $m \times 1$ random vectors a and b,
$$\|a+b\| \leq \|a\| + \|b\|.$$

2. (Cauchy-Schwarz inequality) For any $m \times 1$ random vectors a and b,
$$|a'b| \leq \|a\| \|b\|.$$

3. (Cauchy-Schwarz inequality for expectations) For any random $m \times n$ matrices X and Y, we have
$$\mathbb{E}\|X'Y\| \leq \left[\mathbb{E}\left(\|X\|^2\right)\right]^{1/2} \left[\mathbb{E}\left(\|Y\|^2\right)\right]^{1/2}.$$

4. (Cauchy-Schwarz inequality for matrices) For any $m \times k$ matrix A and $k \times m$ matrix B, we have
$$\|AB\| \leq \|A\| \|B\|.$$

Writing down the consistency argument

Step 1 We start with the OLS estimator:

$$\begin{aligned}
\widehat{\beta} &= \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} X_t Y_t\right) \\
&= \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} X_t \left(\qquad\qquad\qquad\qquad \right)\right) \\
&= \qquad + \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} \qquad\quad \right)
\end{aligned}$$

Step 2 Since $\{X_t\}_{t=1}^{\infty}$ is ??, we can conclude that $\{X_t X_t'\}_{t=1}^{\infty}$ is ??. Furthermore, $X_{it}X_{jt}$, which is the (i,j)th element of $X_t X_t'$, has finite expected value provided that ??, because, by the ??,

$$\mathbb{E}\left|X_{it}X_{jt}\right| \leq \left[\mathbb{E}\left(X_{jt}^2\right)\right]^{1/2}\left[\mathbb{E}\left(X_{it}^2\right)\right]^{1/2} \leq C_1.$$

By the ??, we can conclude that

$$\frac{1}{n}\sum_{t=1}^{n} X_t X_t' \xrightarrow{p} \qquad\quad .$$

Step 3 Let A be a square matrix and $g(A) = \qquad$, provided that A is invertible. By ??, we can conclude that

$$\left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1} \xrightarrow{p} \qquad\quad .$$

Step 4 Next, recall that by definition, $u_t = Y_t - X_t'\beta^*$. Focus on $X_t u_t$. Given that $\{(Y_t, X_t')\}_{t=1}^{\infty}$ is ??, we can conclude that $\{ \qquad \}_{t=1}^{\infty}$ is ??. Furthermore, $X_{jt}u_t$, which is the jth element of $X_t u_t$, has finite expected value provided that X_{jt} and u_t have finite second moments for all j, because, by the ??,

$$\mathbb{E}\left|X_{jt}u_t\right| \leq \left[\mathbb{E}\left(\qquad \right)\right]^{1/2}\left[\mathbb{E}\left(\qquad \right)\right]^{1/2} \leq C_2.$$

By the ??, we can conclude that

$$\frac{1}{n}\sum_{t=1}^{n} X_t u_t \xrightarrow{p} \qquad\quad .$$

Step 5 We can now say that $\mathbb{E}(X_t u_t) =$?? because ??. Now consider the function $g(c, A, B) = c + A^{-1}B$. By ??, we can conclude that

$$\widehat{\beta} \xrightarrow{p}$$

Take a step back.

1. How will the proof change if you have correct specification? Did the proof rely on correct specification?
2. What will happen when $\frac{1}{n}\sum_{t=1}^{n} X_t X_t'$ is not invertible for finite n?
3. What happens in the case of conditional homoscedasticity? How will the argument change?

Writing down the asymptotic normality argument

Step 1 We start once again with the OLS estimator:

$$\widehat{\beta} - \beta^* = \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1} \left(\frac{1}{n}\sum_{t=1}^{n} \qquad \right)$$

$$\sqrt{n}\left(\widehat{\beta} - \beta^*\right) = \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1} \left(\qquad \sum_{t=1}^{n} \qquad \right)$$

Step 2 Show that

$$\left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1} \xrightarrow{p} \left(\mathbb{E}\left(X_t X_t'\right)\right)^{-1}.$$

Step 3 Show that $\{X_t u_t\}_{t=1}^{\infty}$ is ergodic stationary. Next, there are two cases to consider, depending on which chapter you are in.

1. (MDS case) If $\{X_t u_t\}$ is MDS, we have a CLT available (which CLT, and are the conditions satisfied?). As $n \to \infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n} Z_t\right) \xrightarrow{d} N(0, V),$$

   where V =??. Take note that u_t is effectively ε_t here. Refer to our slides for the discussion.

2. (non-MDS case) If $\{X_t u_t\}$ is a zero-mean covariance stationary process, we have a CLT available. In Chapter 6, the CLT is directly assumed to hold for $\{X_t u_t\}$. Therefore, as $n \to \infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n} Z_t\right) \xrightarrow{d} N(0, V),$$

   where V is the long-run variance $\sum_{j=-\infty}^{\infty} \Gamma(j)$ (What is $\Gamma(j)$?)

Step 4 Now, consider the function $g(A, B) = A^{-1}B$. By ?? and ?? property of the multivariate normal:

1. we can conclude that in the IID setting (Chapter 4), we have

$$\sqrt{n}\left(\widehat{\beta} - \beta^*\right) \xrightarrow{d}$$

2. we can conclude that in the ergodic stationary MDS setting (Chapter 5), we have

$$\sqrt{n}\left(\widehat{\beta} - \beta^o\right) \xrightarrow{d}$$

3. we can conclude that in the ergodic stationary non-MDS setting (Chapter 6), we have

$$\sqrt{n}\left(\widehat{\beta} - \beta^*\right) \xrightarrow{d}$$

Take a step back.

1. How will the proof change if you have correct specification? Did the proof rely on correct specification?
2. What will happen when $\dfrac{1}{n}\displaystyle\sum_{t=1}^{n} X_t X_t'$ is not invertible for finite n?
3. What happens in the case of conditional homoscedasticity? How will the argument change?

Writing down the argument for consistent covariance matrix estimation (inspired by Hansen's lecture notes) The main textbook only proves the consistency of

$$\widehat{V}=\frac{1}{n}\sum_{t=1}^{n}X_tX_t'\widehat{u}_t^2$$

for $V=\mathbb{E}\left(X_tX_t'u_t^2\right)$ in Chapters 4 and 5. In Chapter 6, the consistency of the estimator for the long-run covariance matrix is directly assumed rather than proven. We now write down a version of the argument using norms instead of summations. The version using summations can be found in the main textbook.

Step 1 First, we can split up $\widehat{V}$ into two parts, i.e.,

$$\widehat{V}=\frac{1}{n}\sum_{t=1}^{n}X_tX_t'\widehat{u}_t^2=\frac{1}{n}\sum_{t=1}^{n}X_tX_t'\left(u_t^2+\widehat{u}_t^2-u_t^2\right)=\frac{1}{n}\sum_{t=1}^{n}X_tX_t'u_t^2+\frac{1}{n}\sum_{t=1}^{n}X_tX_t'\left(\widehat{u}_t^2-u_t^2\right)$$

Step 2 Let us analyze the first term. We have to ensure that

$$\frac{1}{n}\sum_{t=1}^{n}X_tX_t'u_t^2\xrightarrow{p}\mathbb{E}\left(X_tX_t'u_t^2\right)=V.$$

Since $\{(Y_t,X_t')\}_{t=1}^{\infty}$ is ??, we can conclude that $\{X_tX_t'u_t^2\}_{t=1}^{\infty}$ is also ??. We now have to ensure that $\mathbb{E}\left(\|X_tX_t'u_t^2\|\right)$ is finite. Note that (justify each step, please!)

$$\begin{aligned}\mathbb{E}\left(\left\|X_tX_t'u_t^2\right\|\right)=\mathbb{E}\left(\|X_tX_t'\|\,u_t^2\right)&\leq\left[\mathbb{E}\left(\|X_tX_t'\|^2\right)\right]^{1/2}\left[\mathbb{E}\left(u_t^4\right)\right]^{1/2}\\&\leq\left[\mathbb{E}\left(\|X_t\|^2\,\|X_t'\|^2\right)\right]^{1/2}\left[\mathbb{E}\left(u_t^4\right)\right]^{1/2}\\&=\left[\mathbb{E}\left(\|X_t\|^4\right)\right]^{1/2}\left[\mathbb{E}\left(u_t^4\right)\right]^{1/2}.\end{aligned}$$

Provided that X_{jt} have ?? for all j and u_t have ??, then we can conclude that $\mathbb{E}\left(\|X_tX_t'u_t^2\|\right)<\infty$. By ??, we can conclude that

$$\frac{1}{n}\sum_{t=1}^{n}X_tX_t'u_t^2\xrightarrow{p}\qquad.$$

Step 3 Next, we look at the behavior of the scalar $\widehat{u}_t^2 - u_t^2$, i.e.,

$$\begin{aligned}
\widehat{u}_t &= Y_t - X_t'\widehat{\beta} \\
&= Y_t - X_t'\beta^* + X_t'\beta^* - X_t'\widehat{\beta} \\
\Rightarrow \widehat{u}_t^2 &= u_t^2 - 2\left(\widehat{\beta} - \beta^*\right)' X_t u_t + \left[X_t'\left(\widehat{\beta} - \beta^*\right)\right]^2.
\end{aligned}$$

Our target is to derive an upper bound for the norm of

$$\frac{1}{n}\sum_{t=1}^{n} X_t X_t' \left(\widehat{u}_t^2 - u_t^2\right)$$

and show that this upper bound converges in probability to zero. If this is the case, we can conclude that $\widehat{V} \xrightarrow{p} V$ (Why?). Now, (justify each step, please!)

$$\begin{aligned}
\left\|\frac{1}{n}\sum_{t=1}^{n} X_t X_t' \left(\widehat{u}_t^2 - u_t^2\right)\right\| &\leq \frac{1}{n}\sum_{t=1}^{n} \left\|X_t X_t' \left(\widehat{u}_t^2 - u_t^2\right)\right\| \\
&= \frac{1}{n}\sum_{t=1}^{n} \left\|X_t X_t'\right\| \left|\widehat{u}_t^2 - u_t^2\right| \\
&\leq \frac{1}{n}\sum_{t=1}^{n} \left\|X_t\right\| \left\|X_t'\right\| \left|\widehat{u}_t^2 - u_t^2\right| \\
&= \frac{1}{n}\sum_{t=1}^{n} \left\|X_t\right\|^2 \left|\widehat{u}_t^2 - u_t^2\right| \\
&\leq \underbrace{\frac{1}{n}\sum_{t=1}^{n} \left\|X_t\right\|^2 \left|2\left(\widehat{\beta} - \beta^*\right)' X_t u_t\right|}_{(i)} + \underbrace{\frac{1}{n}\sum_{t=1}^{n} \left\|X_t\right\|^2 \left[X_t'\left(\widehat{\beta} - \beta^*\right)\right]^2}_{(ii)}
\end{aligned}$$

Step 4 First, take a look at (i), justifying each step:

$$\frac{1}{n}\sum_{t=1}^{n} \left\|X_t\right\|^2 \left|2\left(\widehat{\beta} - \beta^*\right)' X_t u_t\right| \leq 2\cdot\frac{1}{n}\sum_{t=1}^{n} \left\|X_t\right\|^2 \left\|\widehat{\beta} - \beta^*\right\| \left\|X_t\right\| \left|u_t\right| = 2\cdot\left(\frac{1}{n}\sum_{t=1}^{n} \left\|X_t\right\|^3 \left|u_t\right|\right) \left\|\widehat{\beta} - \beta^*\right\|.$$

We have to show that the upper bound converges in probability to zero.

1. Since $\widehat{\beta} \xrightarrow{p} \beta^*$, this means that $\widehat{\beta} - \beta^* \xrightarrow{p}$. Consider the function $g(A) =$??. By ??, we can conclude that $\left\|\widehat{\beta} - \beta^*\right\| \xrightarrow{p} 0$.

2. Now we have to ensure that the sample average

$$\frac{1}{n}\sum_{t=1}^{n}\|X_t\|^3|u_t|$$

converges in probability to a finite constant. Note that (justify each step)

$$\mathbb{E}\left[\|X_t\|^3|u_t|\right]=\mathbb{E}\left[\|X_t\|^2\left(\|X_t\||u_t|\right)\right]\leq\left[\mathbb{E}\left(\|X_t\|^4\right)\right]^{1/2}\left[\mathbb{E}\left(\|X_t\|^2|u_t|^2\right)\right]^{1/2}.$$

Since $\{X_t'\}$ is ??, $\left\{\|X_t\|^4\right\}$ is ??. Provided that X_{jt} has finite fourth moments for all j, $\mathbb{E}\left(\|X_t\|^4\right)<\infty$.

These previous two points allow us to conclude that

$$\left(\frac{1}{n}\sum_{t=1}^{n}\|X_t\|^3|u_t|\right)\left\|\widehat{\beta}-\beta^*\right\|\xrightarrow{p}\qquad.$$

Step 5 Second, take at a look at (ii), justifying each step:

$$\frac{1}{n}\sum_{t=1}^{n}\|X_t\|^2\left[X_t'\left(\widehat{\beta}-\beta^*\right)\right]^2\leq\frac{1}{n}\sum_{t=1}^{n}\|X_t\|^2\|X_t'\|^2\left\|\widehat{\beta}-\beta^*\right\|^2=\left(\frac{1}{n}\sum_{t=1}^{n}\|X_t\|^4\right)\left\|\widehat{\beta}-\beta^*\right\|^2.$$

Just like in Step 4, we have to show that the upper bound converges in probability to zero.

1. Since $\widehat{\beta}\xrightarrow{p}\beta^*$, this means that $\widehat{\beta}-\beta^*\xrightarrow{p}\qquad$. Consider the function $g(A)=$??. By ??, we can conclude that $\left\|\widehat{\beta}-\beta^*\right\|^2\xrightarrow{p}0$.
2. Now we have to ensure that the sample average

$$\frac{1}{n}\sum_{t=1}^{n}\|X_t\|^4$$

converges in probability to a finite constant. Show this.

These previous two points allow us to conclude that

$$\left(\frac{1}{n}\sum_{t=1}^{n}\|X_t\|^4\right)\left\|\widehat{\beta}-\beta^*\right\|^2\xrightarrow{p}\qquad.$$

Step 6 Taking Steps 4 and 5 together, we can conclude that

$$\left\|\frac{1}{n}\sum_{t=1}^{n}X_tX_t'\left(\widehat{u}_t^2-u_t^2\right)\right\|\xrightarrow{p}\qquad.$$

Therefore, $\widehat{V} \xrightarrow{p} V$.

Finally, Q can be consistently estimated by repeating the argument we used in the consistency proof. In particular, we have

$$\frac{1}{n}\sum_{t=1}^{n} X_t X_t' \xrightarrow{p} \qquad .$$

Consider the function $g(A, B) = A^{-1}BA^{-1}$. By ??, we can conclude that

$$\left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1} \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t' \widehat{u}_t^2\right) \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1} \xrightarrow{p} \qquad .$$

Take a step back.

1. Think carefully why we cannot apply the argument in Step 2 for the sequence $\{X_t X_t' \widehat{u}_t^2\}$.
2. How will the proof change if you have correct specification? Did the proof rely on correct specification?
3. What will happen when $\frac{1}{n}\sum_{t=1}^{n} X_t X_t'$ is not invertible for finite n?
4. What happens in the case of conditional homoscedasticity? How will the argument change?